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TESTING THE ACCURACY OF A THREE-  
DIMENSIONAL ACOUSTIC COUPLED MODE MODEL

by

Hiroyuki Ogawa

December 1991

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TESTING THE ACCURACY OF A  
THREE-DIMENSIONAL ACOUSTIC COUPLED MODE MODEL

by


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Submitted in partial fulfillment of the requirements for the degree of  
MASTER OF SCIENCE IN PHYSICAL OCEANOGRAPHY

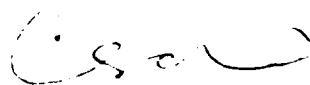
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## ABSTRACT

The three-dimensional coupled mode sound propagation model of Chiu and Ehret (1990) is tested for its accuracy in modeling horizontal sound refraction. This test is achieved by comparing results generated by this model with exact analytic solutions to a parabolic approximation of the Helmholtz Equation (Seigmann et al., 1990). Specifically, the acoustic wavefields associated with two cases of horizontally variable sound speed are considered. In Case I the sound speed varies only with azimuth, while in Case II there is radial and azimuth variation. The acoustic quantities used for the comparison include the modal modulation envelopes and transmission loss. Errors in the mode amplitude and phase are quantified and the significance of horizontal refraction is discussed.



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## LIST OF NOTATION

$A$ :	the coefficient of eigenfunction
$A_p$ :	the amplitude of the envelope function for PE
$C(\vec{r})$ :	the sound speed as a function of position $\vec{r}$
$C_0$ :	the reference sound speed
$D_n, E_n, F_n, \tilde{G}_{mn}, H_{mn}$ :	the coefficients in the equation for normal mode envelope function
$H$ :	ocean depth
$H_0^1(k_0 r)$ :	zeroth order Hankel function of the first kind
$k(\vec{r})$ :	the acoustic wave number
$k_n$ :	the corresponding horizontal wavenumber (eigenvalue)
$n(\vec{r})$ :	index of refraction
$P(\vec{r}, t)$ :	the acoustic pressure
$P_n(r, \theta)$ :	the mode amplitude function at point $(r, \theta)$ for normal mode theory
$\Phi_{NM}$ :	an acoustic pressure for numerical normal mode
$\Phi_{PE}$ :	an acoustic pressure for analytic PE
$q(\vec{r}, t)$ :	an acoustic source
$\vec{r}$ :	position vector of function $(r, \theta, z)$
$r$ :	radial range
$t$ :	time
$TL$ :	transmission loss
$u(r, \theta, z)$ :	modulation on PE
$U_1(r, \theta)$ :	analytic envelope function associated with Mode 1
$U_n$ :	envelope function for normal mode
$U_p$ :	envelope function for PE
$v(r)$ :	rapidly varying portion for PE
$U_n^i$ :	envelope function at the $i^{\text{th}}$ iteration for mode $n$
$f$ :	acoustic frequency
$\hat{r}$ :	the unit directional vectors in $r$

$TL_{NM}$ :	the numerical NM transmission loss
$TL_{PE}$ :	the analytic PE transmission loss
$Z_n(r, \theta)$ :	$n^{\text{th}}$ local normal mode (eigenmode) at point $(r, \theta)$
$\alpha_0, \beta_0$ :	arbitrary constants
$\gamma_j$ :	vertical wavenumber of the $j^{\text{th}}$ mode
$\gamma_{mn}, \beta_{mn}, B_{mn}$ :	mode coupling coefficients
$\Theta$ :	phase of the PE envelope function
$\theta$ :	azimuth angle
$\Phi(\vec{r})$ :	the time-independent acoustic pressure component
$\phi_n$ :	range integral of horizontal wavenumber
$w$ :	circular frequency
$\hat{\theta}$ :	the unit directional vector in $\theta$
$z$ :	depth

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## I INTRODUCTION

The field of three-dimensional (3D) underwater acoustics has been developed in an attempt to describe the properties and characteristics of sound propagation in the heterogeneous ocean environment (Lynch and Chiu, 1989). At present, there are three major approaches used in studying the 3D underwater sound field. One of these, ray theory, is premised upon a high-frequency approximation method. This method gives a geometric-optics solution which involves ray-tracing in a spatially varying sound speed field. The Hamiltonian Acoustic Ray Tracing Program for the Ocean (HARPO) is a versatile numerical code developed by Jones et al. (1986) for the computation of 3D rays. The primary deficiency of this theory is that it cannot adequately address the behavior of low-frequency sound due to the neglect of sound dispersion and diffraction.

The second approach uses a parabolic approximation to the acoustic wave equation, which was introduced by Tappert (1977). A 3D numerical parabolic equation (PE) model was developed by Lee et al. (1988) using an implicit finite difference scheme. Another 3D PE model was arrived at by Baer (1981) utilizing a split-step Fourier algorithm. In a recent study, analytic solutions to the 3D parabolic equation were obtained by Seigmann et al., (1990). These solutions are valuable for testing the accuracy of 3D numerical models.

A third approach is the normal mode method. Pierce presented a three-dimensional version of this method in 1965. The normal mode method is based upon a separable solution to the wave equation. Pierce assumed an adiabatic acoustic environment which leads to the neglect of coupling

presented by Chiu and Ehret (1990). This later 3D normal mode model accounts for both horizontal refraction and model coupling.

Any improvement in acoustic modeling must be quantified. We shall endeavor to test the accuracy of the 3-D coupled normal mode model of Chiu and Ehret (1990) in regard to horizontal refraction.

To examine the accuracy of the Chiu-Ehret model, we compare the results of their model with those from analytic solutions to the parabolic equation arrived at by Seigmann, et al. (1990). The results of the Chiu-Ehret model and the analytic solutions of the parabolic equation are compared for two cases of horizontal variation in sound speed. As a means of comparing the results, we examine the slow variations of the complex pressure envelope function and transmission loss.

Our simulated acoustic field has a range of 100 km with azimuth from  $30^\circ$  to  $180^\circ$ . Two variations in the horizontal sound speed field were used. In Case I the sound speed varies only with the azimuth angle while Case II varies both with the azimuth and radial.

The normal mode theory used by Chiu and Ehret is described in detail in the Appendix. The appendix also includes a description of the PE approximation and the analytic solutions developed by Seigmann et al. (1990).

## II. ANALYSIS AND METHOD OF MODEL COMPARISON

The three-dimensional coupled normal mode model was applied to two sound speed fields for which exact analytic solutions to the PE approximation are available. These analytic PE solutions for vertically invariant sound speed were developed by Seigmann et al. (1990). Their development was based on expressing the "time-independent" acoustic pressure component as

$$\Phi_{PE}(r, \theta, z) = U_p(r, \theta) \sin(\gamma_j z) \exp \left\{ -\frac{ik_0 r}{2} \left( \frac{\gamma_j}{k_0} \right)^2 \left[ 1 - \frac{1}{4} \left( \frac{\gamma_j}{k_0} \right)^2 \right] \right\} \times \left( \frac{2}{\pi k_0 r} \right)^{\frac{1}{2}} \exp \left[ i \left( k_0 r - \frac{\pi}{4} \right) \right]$$

where

$$k_0 = \frac{2\pi f}{C_0}$$

$$\gamma_j = \left[ j + \frac{1}{2} \right] \frac{\pi}{H} \quad (j = 0, 1, 2, \dots)$$

$C_0$  = reference sound speed

$f$  = acoustic frequency

$H$  = ocean depth.

The reduced complex envelope function

$$U_p(r, \theta) = A_p(r, \theta) \exp \{ i \Theta(r, \theta) \}$$

$U_p$  is governed by

$$\frac{\partial U_p}{\partial r} = \frac{ik_0}{2} \left\{ (n^2 - 1) \left[ 1 + \frac{1}{2} \left( \frac{\gamma_j}{k_0} \right)^2 \right] - \frac{1}{4} (n^2 - 1) - \frac{1}{2} \left( \frac{\gamma_j}{k_0} \right)^4 + \frac{1}{(k_0 r)^2} \frac{\partial^2}{\partial \theta^2} \right\} U_p \quad (2.1)$$

where

$$n(\vec{r}) = \frac{C_0}{C(\vec{r})} \quad (\vec{r} = (r, \theta, z))$$

$C(\vec{r})$  = sound speed as a function of position  $\vec{r}$ .

Substitution and separation of real and imaginary parts gives two coupled equations governing  $A_p$  and  $\Theta$ :

$$\frac{\partial A_p}{\partial r} + \frac{1}{k_0 r^2} \frac{\partial A_p}{\partial \theta} \frac{\partial \Theta}{\partial \theta} + \frac{A_p}{2k_0 r^2} \frac{\partial^2 \Theta}{\partial \theta^2} = 0 \quad (2.2)$$

$$\begin{aligned} & (n^2 - 1) \left[ 1 + \frac{1}{2} \left( \frac{\gamma_j}{k_0} \right)^2 \right] - \frac{1}{4} (n^2 - 1)^2 - \frac{1}{2} \left( \frac{\gamma_j}{k_0} \right)^4 \\ &= \frac{2}{k_0} \frac{\partial \Theta}{\partial r} + \frac{1}{(k_0 r)^2} \left( \frac{\partial \Theta}{\partial \theta} \right)^2 - \frac{1}{(k_0 r)^2} \frac{1}{A_p} \frac{\partial^2 A_p}{\partial \theta^2}. \end{aligned} \quad (2.3)$$

If one chooses  $\Theta(r, \theta) = \alpha(r)\theta$ , equation (2.2) may be solved for the amplitude, i.e.,

$$A_p(r, \theta) = F \left[ \theta - \frac{1}{k_0} \int \frac{\alpha(s)}{s^2} ds \right]. \quad (2.4)$$

Choosing  $\alpha(r)$  and the functional dependence  $F$  results in a reduced envelope function (and ultimately acoustic pressure) while equation (2.3) can be solved for the index of refraction  $n(\vec{r})$  and thus the sound speed field  $C(\vec{r})$ . Equation (2.3) is for a wide angle PE approximation. This wide angle form will be used throughout. To judge the accuracy of the Chiu-Ehret model in calculating horizontal refraction we compare the numerical coupled mode results with the analytic PE results for two sound speed fields or two functional definitions of  $\alpha(r)$ .

## A. INDEX OF REFRACTION

### 1. Case I

For Case 1 we choose  $\alpha(r) = \alpha_0 k_0 r$ , or

$$\Theta = \alpha_0 k_0 r \theta \quad (2.5)$$

where  $\alpha_0$  is an arbitrary constant. From equation (2.4) the amplitude is

$$A_p(r, \theta) = F(\theta - \alpha_0 \ln r).$$

Let the function  $F$  be multiplied by  $\beta_0 k_0$ . The amplitude has the final form

$$A_p(r, \theta) = \beta_0 k_0 (\theta - \alpha_0 \ln r) \quad (2.6)$$

where  $\beta_0$  is an arbitrary constant with units of length. The square of the index of refraction from (2.5) and (2.6) is

$$n^2 = 1 + 2 \left[ 1 + \frac{1}{2} \left( \frac{\gamma_1}{k_0} \right)^2 \right] - \left\{ 4 \left[ 1 + \frac{1}{2} \left( \frac{\gamma_1}{k_0} \right)^2 \right]^2 - 2 \left[ 4 \alpha_0 \theta + 2 \alpha_0^2 + \left( \frac{\gamma_1}{k_0} \right)^4 \right] \right\}^{\frac{1}{2}}. \quad (2.7)$$

The resultant sound speed is shown in Figure 1 for  $\alpha_0 = 0.005$  and  $C_0 = 1500$  m/s.

### 2. Case II

For Case II we choose

$$\Theta = \alpha_0 k_0^2 r^2 \theta \quad (2.8)$$

and hence

$$A_p(r, \theta) = \beta_0 k_0 (\theta - \alpha_0 k_0 r). \quad (2.9)$$

The square of the index of refraction is now

$$n^2 = 1 + 2 \left[ 1 + \frac{1}{2} \left( \frac{\gamma_1}{k_0} \right)^2 \right] - \left\{ 4 \left[ 1 + \frac{1}{2} \left( \frac{\gamma_1}{k_0} \right)^2 \right]^2 - 2 \left[ 8\alpha_0 k_0 r \theta + 2\alpha_0^2 k_0^2 r^2 + \left( \frac{\gamma_1}{k_0} \right)^4 \right] \right\}^{\frac{1}{2}}. \quad (2.10)$$

The resulting sound speed is shown in Figure 2 for  $\alpha_0 = 4 \times 10^{-7}$  and  $C_0 = 1500$  m/s.

## B. ENVELOPE FUNCTION

We now have two analytic sound speed fields and the associated acoustic pressure fields derived from the parabolic equation approximation. Considering only  $j = 1$ , Seigmann et al., (1990) define pressure as

$$\Phi_{PE}(r, \theta, z) = \left( \frac{2}{\pi k_0 r} \right)^{\frac{1}{2}} e^{i \left( k_0 r - \frac{\pi}{4} \right)} A_p e^{i\theta} \sin(\gamma_1 z) \exp \left\{ -\frac{i k_0 r}{2} \left( \frac{\gamma_1}{k_0} \right)^2 \left[ 1 - \frac{1}{4} \left( \frac{\gamma_1}{k_0} \right)^2 \right] \right\} \quad (2.11)$$

For Case I, using the amplitude and phase expressed in (2.6) and (2.5), respectively, we obtain

$$\begin{aligned} \Phi_{PE}(r, \theta, z) = & \left( \frac{2}{\pi k_0 r} \right)^{\frac{1}{2}} e^{i \left( k_0 r - \frac{\pi}{4} \right)} \beta_0 k_0 (\theta - \alpha_0 \ln r) e^{i \alpha_0 k_0 r \theta} \sin(\gamma_1 z) \\ & \times \exp \left\{ -\frac{i k_0 r}{2} \left( \frac{\gamma_1}{k_0} \right)^2 \left[ 1 - \frac{1}{4} \left( \frac{\gamma_1}{k_0} \right)^2 \right] \right\} \end{aligned} \quad (2.12)$$

For Case II, using (2.8) and (2.9), we get

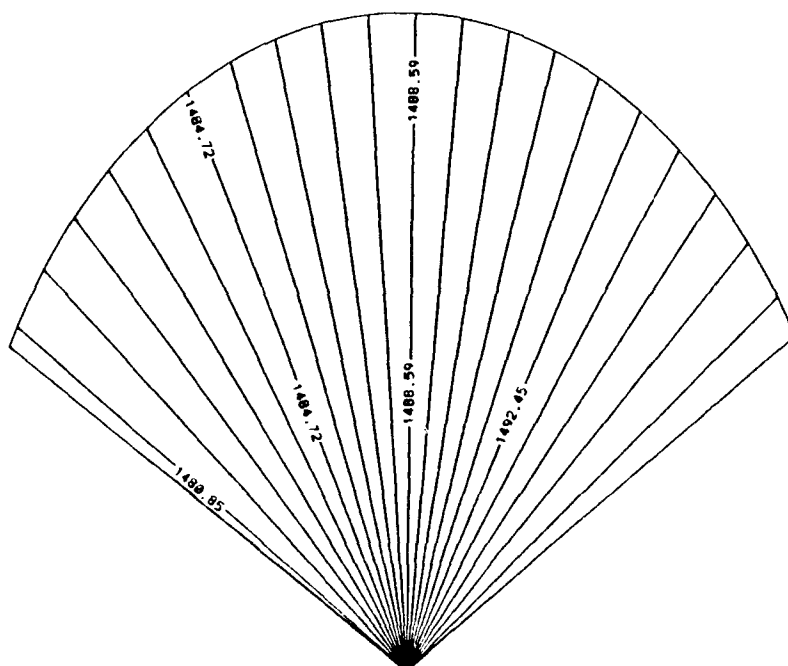


Figure 1. Case I Sound Speed Field for  $\alpha_0 = 0.005$  and  $C_0 = 1500$  m/s

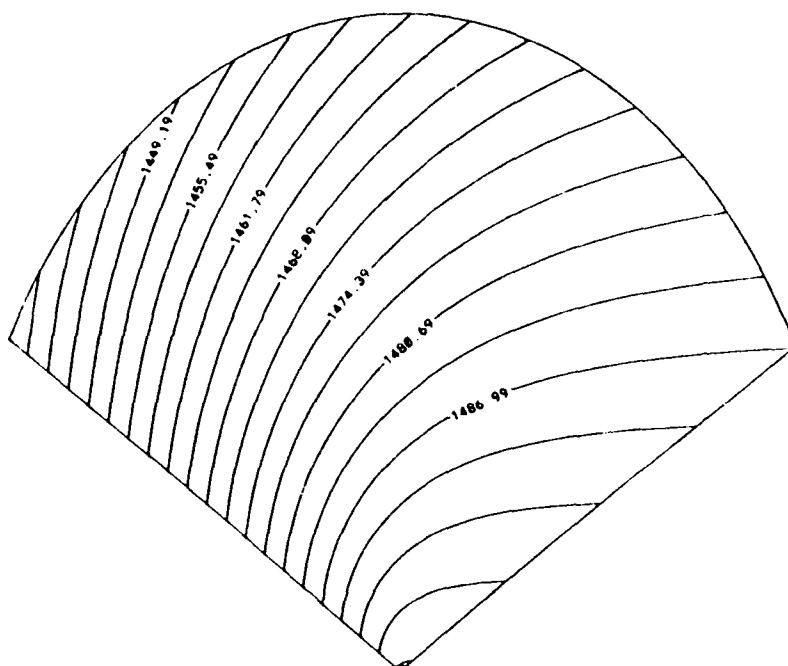


Figure 2. Case II Sound Speed Field for  $\alpha_0 = 4 \times 10^{-7}$  and  $C_0 = 1500$  m/s

$$\begin{aligned}\Phi_{PE}(r, \theta, z) = & \left( \frac{2}{\pi k_0 r} \right)^{\frac{1}{2}} e^{i \left( k_0 r - \frac{\pi}{4} \right)} \beta_0 k_0 (\theta - \alpha_0 k_0 r) e^{i \alpha_0 k_0^2 r^2 \theta} \sin(\gamma_1 z) \\ & \times \exp \left\{ -\frac{i k_0 r}{2} \left( \frac{\gamma_1}{k_0} \right)^2 \left[ 1 - \frac{1}{4} \left( \frac{\gamma_1}{k_0} \right)^2 \right] \right\}.\end{aligned}\quad (2.13)$$

In terms of normal modes, pressure can be expressed as

$$\Phi_{NM}(r, \theta, z) = \frac{1}{\sqrt{r}} \sum_n U_n(r, \theta) e^{i \int_0^r k_n(r, \theta) dr} Z_n(z; r, \theta) \quad (2.14)$$

where  $Z_n$  is the  $n^{th}$  normal mode (eigenmode),  $k_n$  is the corresponding horizontal wavenumber (eigenvalue) and  $U_n$  is the corresponding slowly varying modulation envelope. See the appendix for details.

Since we are considering depth-invariant sound speed fields, an analytic solution is available for the eigenvalues and eigenmodes. The first mode is

$$Z_1 = \sqrt{\frac{2}{H}} \sin(\gamma_1 z) \quad \text{where} \quad \gamma_1 = \sqrt{k^2 - k_1^2} = \frac{\pi}{2H}$$

and thus, keeping only the contribution of the first mode, (2.14) becomes

$$\Phi_{NM}(r, \theta, z) = \sqrt{\frac{2}{H}} \frac{1}{\sqrt{r}} U_1(r, \theta) e^{i \int_0^r k_1(r, \theta) dr} \sin(\gamma_1 z). \quad (2.15)$$

The normal mode model entails a numerical calculation of the slowly varying envelope function  $U_n$ . Therefore, the quantification of model accuracy can be achieved by comparing the numerical  $U_n$  to those derived from the analytic PE solutions. The analytic envelope function  $U_1(r, \theta)$  for Case I is derived by equating (2.12) to (2.15). The resulting expression is

$$U_1(r, \theta) = \sqrt{\frac{H}{\pi k_0}} \beta_0 k_0 (\theta - \alpha_0 \ln r) \exp i \left\{ k_0 r \left[ 1 - \frac{1}{2} \left( \frac{\gamma_1}{k_0} \right)^2 \left( 1 - \frac{1}{4} \left( \frac{\gamma_1}{k_0} \right)^2 \right) + \alpha_0 \theta \right] - \int_0^r k_1 dr - \frac{\pi}{4} \right\}^{\frac{1}{2}}. \quad (2.16)$$

where

$$\begin{aligned} k_1^2 &= k^2 - \gamma_1^2 \\ &= k_0^2 \left[ n^2 - \left( \frac{\gamma_1}{k_0} \right)^2 \right]. \end{aligned}$$

With the index of refraction  $n$  given by (2.7), the range integral of  $k_1$  can be expressed as

$$\int_0^r k_1 dr = k_0 r \left\{ 1 + 2 \left[ 1 + \frac{1}{2} \left( \frac{\gamma_1}{k_0} \right)^2 \right] - \sqrt{4 \left[ 1 + \frac{1}{2} \left( \frac{\gamma_1}{k_0} \right)^2 \right]^2 - 2 \left[ 4 \alpha_0 \theta + 2 \alpha_0^2 + \left( \frac{\gamma_1}{k_0} \right)^4 \right]} - \left( \frac{\gamma_1}{k_0} \right)^2 \right\}^{\frac{1}{2}}.$$

It follows that the analytic envelope function  $U_1(r, \theta)$  can be recast as

$$U_1(r, \theta) = \sqrt{\frac{H}{\pi k_0}} \beta_0 k_0 (\theta - \alpha_0 \ln r) e^{i \left( k_0 r \phi - \frac{\pi}{4} \right)} \quad (2.17)$$

where

$$\phi = 1 - \frac{1}{2} \left( \frac{\gamma_1}{k_0} \right)^2 \left[ 1 - \frac{1}{4} \left( \frac{\gamma_1}{k_0} \right)^2 \right] + \alpha_0 \theta - \left\{ 3 - \sqrt{4 \left[ 1 + \frac{1}{2} \left( \frac{\gamma_1}{k_0} \right)^2 \right]^2 - 2 \left[ 4 \alpha_0 \theta + 2 \alpha_0^2 + \left( \frac{\gamma_1}{k_0} \right)^4 \right]} \right\}^{\frac{1}{2}}$$

For Case II, the wave number integral, using (2.10), becomes

$$\int_0^r k_1 dr = k_0 r \left\{ \frac{3}{2} - \frac{1}{2} \left[ 1 + \frac{1}{2} \left( \frac{\gamma_1}{k_0} \right)^2 \right]^2 + \frac{1}{4} \left( \frac{\gamma_1}{k_0} \right)^4 + \alpha_0 k_0 r \theta + \frac{1}{6} \alpha_0^2 k_0^2 r^2 \right\}.$$

The analytic envelope function  $U_1$  is then

$$U_1(r, \theta) = \sqrt{\frac{H}{\pi k_0}} \beta_0 k_0 (\theta - \alpha_0 k_0 r) e^{i \left( k_0 r \phi - \frac{\pi}{4} \right)} \quad (2.18)$$

where

$$\begin{aligned} \phi &= 1 - \frac{1}{2} \left( \frac{\gamma_1}{k_0} \right)^2 \left[ 1 - \frac{1}{4} \left( \frac{\gamma_1}{k_0} \right)^2 \right] + \alpha_0 k_0 r \theta - \frac{1}{k_0 r} \int_0^r k_1 dr \\ &= -\frac{1}{6} \alpha_0^2 k_0^2 r^2. \end{aligned}$$

For Case I we use the analytic envelope function (2.17) to define the initial condition at  $r = r_0$  for the numerical model run. The condition is

$$U_1(r_0, \theta) = \sqrt{\frac{H}{\pi k_0}} \beta_0 k_0 (\theta - \alpha_0 \ln r_0) e^{i \left( k_0 r_0 \phi - \frac{\pi}{4} \right)} \quad (2.19)$$

where

$$\begin{aligned} \phi &= 1 - \frac{1}{2} \left( \frac{\gamma_1}{k_0} \right)^2 \left[ 1 - \frac{1}{4} \left( \frac{\gamma_1}{k_0} \right)^2 \right] \\ &+ \alpha_0 \theta - \left\{ 3 - \sqrt{4 \left[ 1 + \frac{1}{2} \left( \frac{\gamma_1}{k_0} \right)^2 \right]^2 - 2 \left[ 4 \alpha_0 \theta + 2 \alpha_0^2 + \left( \frac{\gamma_1}{k_0} \right)^4 \right]} \right\}^{\frac{1}{2}}. \end{aligned}$$

For Case II we use the analytic envelope function from (2.18) at  $r = r_0$ . The Case II initial condition is

$$U_1(r_0, \theta) = \sqrt{\frac{H}{\pi k_0}} \beta_0 k_0 (\theta - \alpha_0 k_0 r_0) e^{i\left(k_0 r_0 \phi - \frac{\pi}{4}\right)} \quad (2.20)$$

where

$$\phi = -\frac{1}{6} \alpha_0^2 k_0^2 r_0^2.$$

### III. MODEL RUNS

Our computational domain is in a cylindrical coordinate system. The horizontal range is from 1 km to 100 km. The azimuth range is 30° to 150°. The depth is 4000 m. The domain is the same for both cases.

We use a frequency of 50 Hz and source depth of 1000 m. The reference sound speed is 1500 m/s. We consider mode 1 only. The vertical boundary conditions for the problem are pressure release surface and rigid bottom. Two range and azimuth dependent sound speed fields as presented in the last chapter are used. For Case I sound speed ranges from 1481 m/s to 1496 m/s, varying only in azimuth with a gradient of 0.128 to 0.130 m/s/degree (see Table 1). For Case II sound speed varies in both range and azimuth (see Table 1 again). Here sound speed goes from 1437 m/s to 1500 m/s with a radial gradient of 0.13 to 0.657 m/s/km and an azimuth gradient of 0.004 to 0.431 m/s/degree. Both selections of sound speed fields contain realistic gradients as observed in the ocean.

TABLE 1. THE AZIMUTH AND RADIAL SOUND SPEED VARIATION

Case	Range r[km]	Sound Speed [m/s]			Azimuth Variation $\frac{\partial C}{\partial \theta} \left[ \frac{m}{s/deg} \right]$			Radial Variation $\frac{\partial C}{\partial r} \left[ \frac{m}{s/km} \right]$		
		$\theta=30^\circ$	$\theta=90^\circ$	$\theta=150^\circ$	$\theta=30^\circ$	$\theta=90^\circ$	$\theta=150^\circ$	$\theta=30^\circ$	$\theta=90^\circ$	$\theta=150^\circ$
I	1	1496	1488	1481	0.1303	0.1289	0.1275	0	0	0
	50	1496	1488	1481	0.1303	0.1289	0.1275	0	0	0
	100	1496	1488	1481	0.1303	0.1289	0.1275	0	0	0
II	1	1500	1500	1499	0.0044	0.0043	0.0043	0.1316	0.3944	0.6482
	50	1493	1480	1468	0.2217	0.2178	0.2142	0.1310	0.3848	0.6221
	100	1487	1462	1437	0.4265	0.4127	0.4001	0.1301	0.3763	0.5992

Eigenmodes and eigenvalues are calculated using finite difference approximations to (A.8) in the appendix with a 40 m vertical grid spacing and a matrix eigenvalue solver.

To obtain the numerical envelope function a first order differential equation ((A.24) in the appendix) is integrated. The radial integration step size is 0.5 km. The 61 integration paths were separated by two degrees.

## IV. RESULTS

### A. CASE I

In comparing the numerical normal mode (NM) and analytic parabolic equation (PE) results we focus on the slowly varying modulating pressure envelope. The removal of the rapid oscillations, which have wave lengths of order  $2\pi/k_0$ , makes the displays of results easier to interpret.

Figures 3 and 4 show the amplitude and phase, respectively, of the envelope function calculated by the normal mode model using the Case I sound speed field. Figures 5 and 6 are the amplitude and phase, respectively, of the envelope based on the analytic solution to the parabolic equation. Figure 7 shows the relative difference, defined as

$$\frac{U_{PE}(\text{analytic solution}) - U_{NM}(\text{numerical solution})}{U_{PE}(\text{analytic solution})}$$

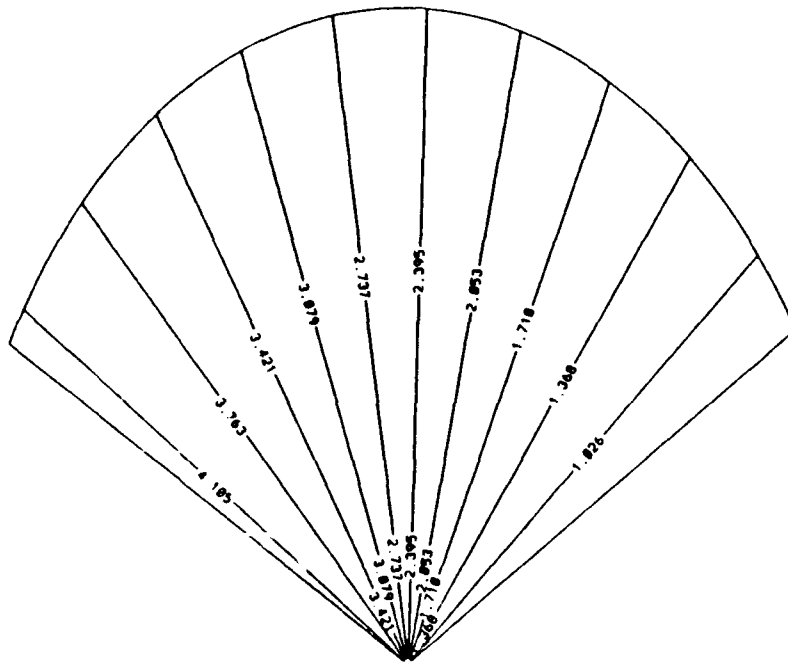
The percent difference is everywhere less than 5%. The difference in NM and PE phase is shown in Figure 8.

Transmission loss at a depth of 1000 m calculated by the NM model,  $TL_{NM}$ , is shown in Figure 9. The magnitude of the analytic PE pressure, from (2.12), is

$$|P| = \sqrt{\frac{2}{\pi k_0 r}} A_p \sin(\gamma_j z). \quad (3.1)$$

Substituting (2.6) in (3.1), we get

$$|P| = \sqrt{\frac{2}{\pi k_0 r}} \beta_0 k_0 (\theta - \alpha_0 \ln r) \sin\left[\frac{\pi}{2H} z\right].$$



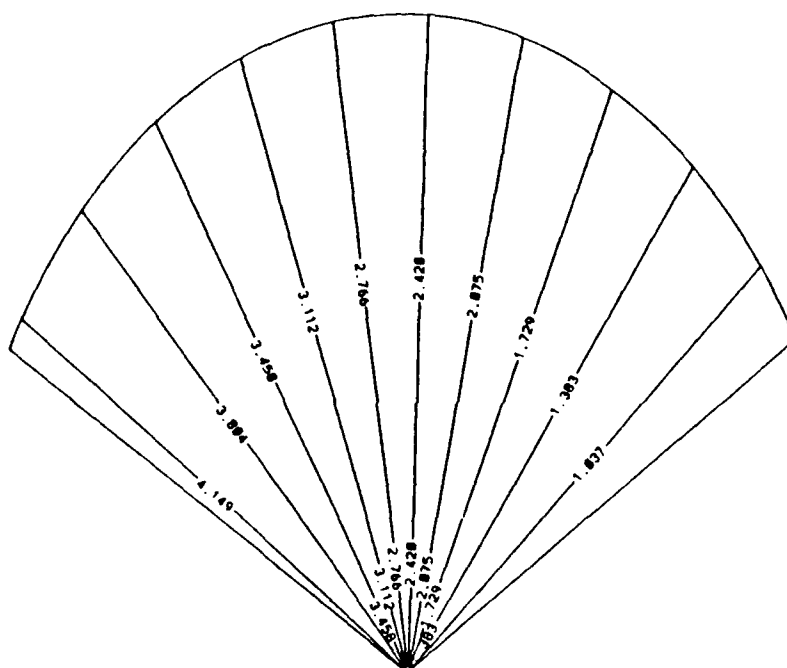


Figure 5. Amplitude of  $U_{pE}$  for Case I with  $\alpha = 0.005$  and  $\beta = 0.1$

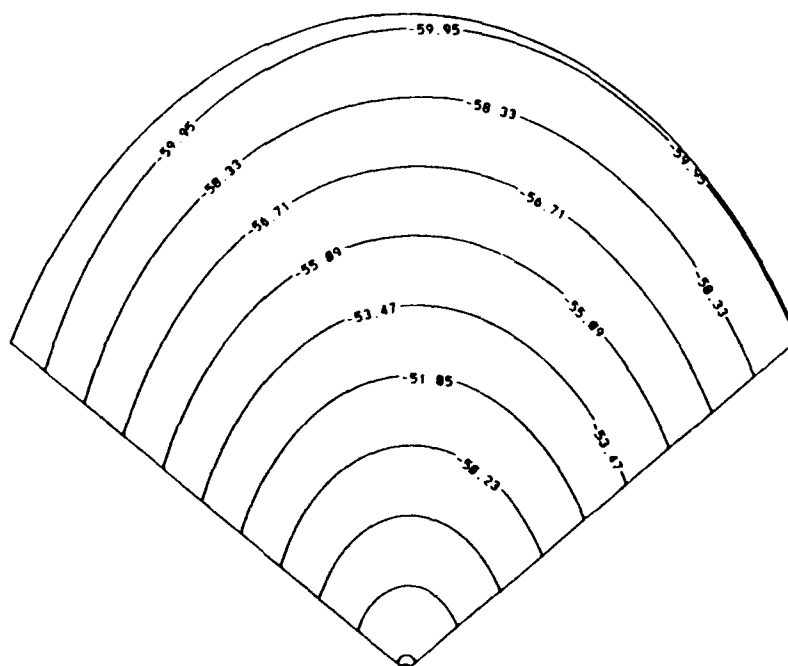


Figure 6. Phase of  $U_{pE}$  for Case I with  $\alpha = 0.005$  and  $\beta = 0.1$

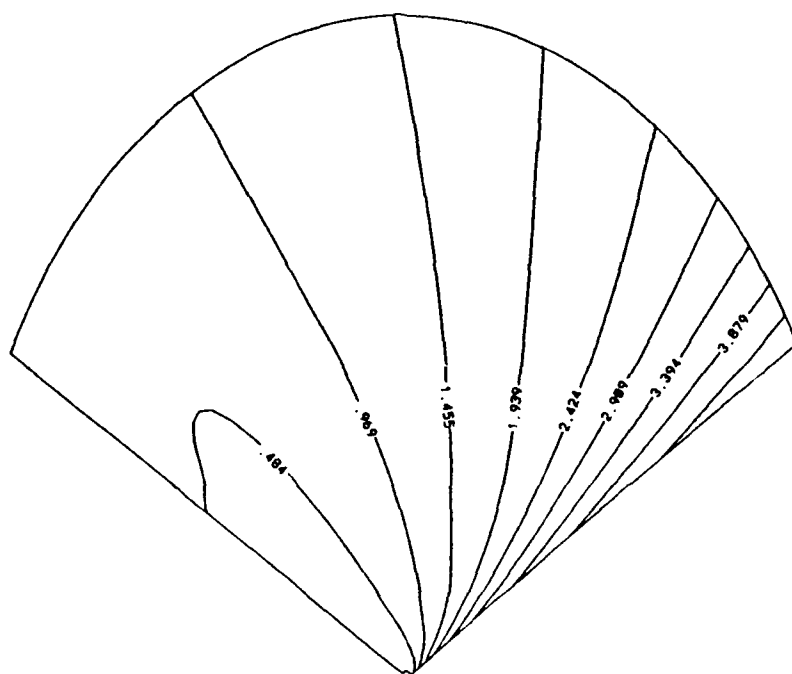


Figure 7. Amplitude Difference in Percent for Case I with  $\alpha = 0.005$  and  $\beta = 0.1$

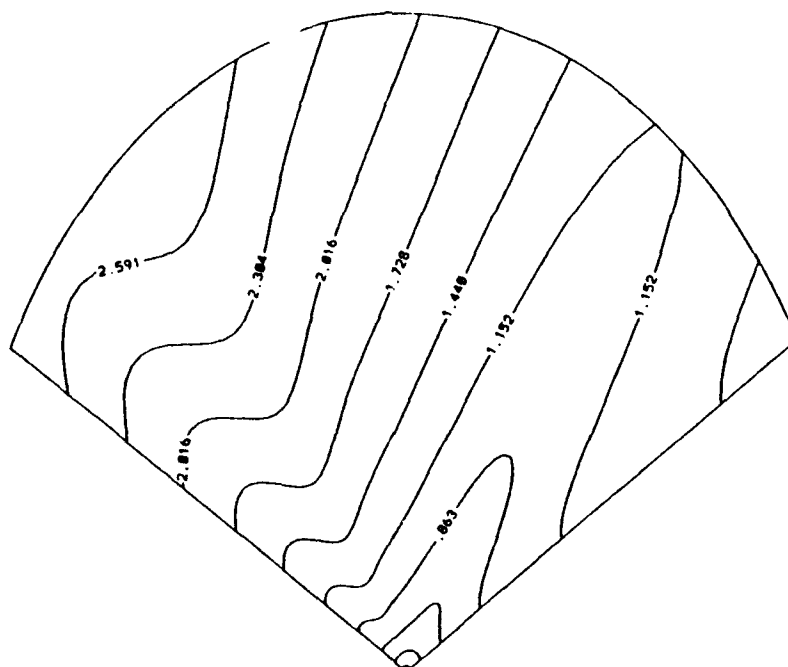


Figure 8. Phase Difference in Degree for Case I with  $\alpha = 0.005$  and  $\beta = 0.1$

The analytic PE transmission loss  $TL_{PE}$  is thus

$$\begin{aligned} TL_{PE} &= -20\log_{10}|P| \\ &= 10\log_{10}\frac{\pi k_0 r}{2} - 20\log_{10}|\beta_0 k_0(\theta - \alpha_0 \ln r)| - 20\log\left|\sin\frac{\pi}{2H}z\right|. \end{aligned}$$

Figure 10 displays  $TL_{PE}$  at a depth of 1000m. Both transmission loss results show that the azimuth variation of sound speed produces lower loss at large azimuth angle.

The difference between the analytic PE transmission loss  $TL_{PE}$  and the numerical NM transmission loss  $TL_{NM}$  is shown in Figure 11. Since the modal transmission loss is a function of amplitude alone this difference has the same shape as the relative error in the amplitude of the envelope function. The difference in transmission loss is everywhere less than 1 dB.

## B. CASE II

In the second case, we examined a sound speed field that varies in range and azimuth, closely representing an eddy or ring structure in the real ocean.

Figures 12 and 13 show the amplitude and phase, respectively, of the envelope function  $U_{NM}$  from the normal mode model. Note that azimuth variation of the amplitude is larger than range variation. Figures 14 and 15 are the amplitude and phase of the envelope function  $U_{PE}$  based on the analytic solution to the wide-angle PE. Figure 16 shows the relative difference of the amplitude. The percent difference is everywhere less than 2%. The difference between the NM and PE phases is shown in Figure 17. The difference of phase is everywhere less than one degree.

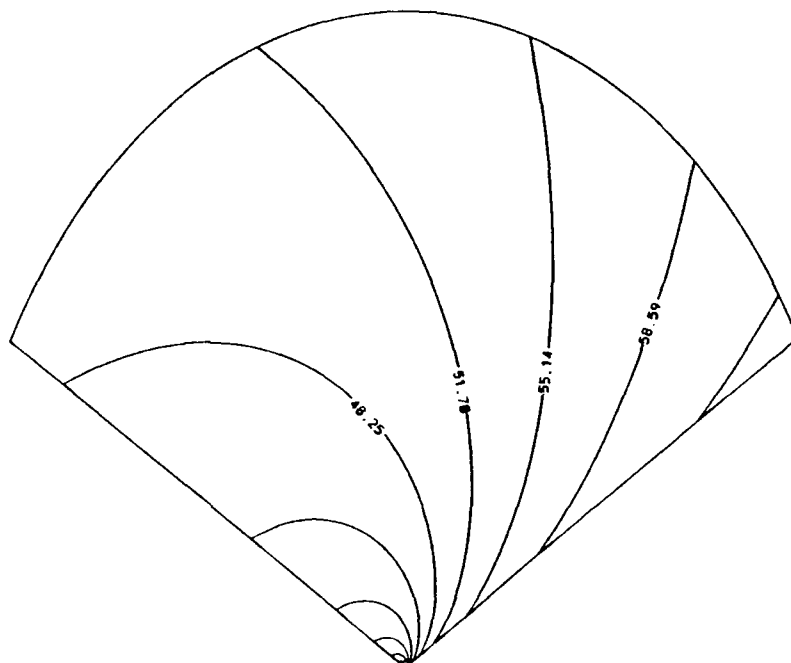


Figure 9.  $TL_{NM}$  at a Depth of 1000 m for Case I with  $\alpha = 0.005$  and  $\beta = 0.1$

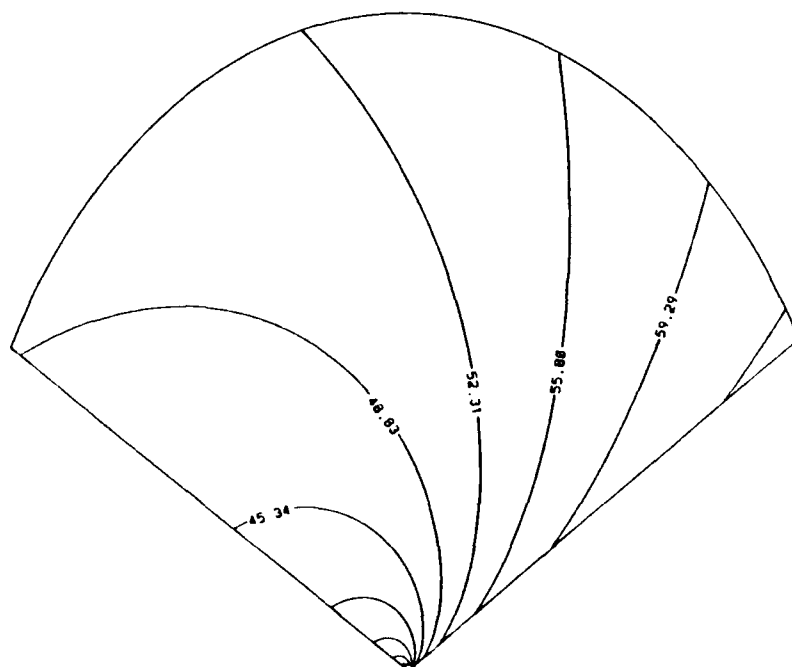
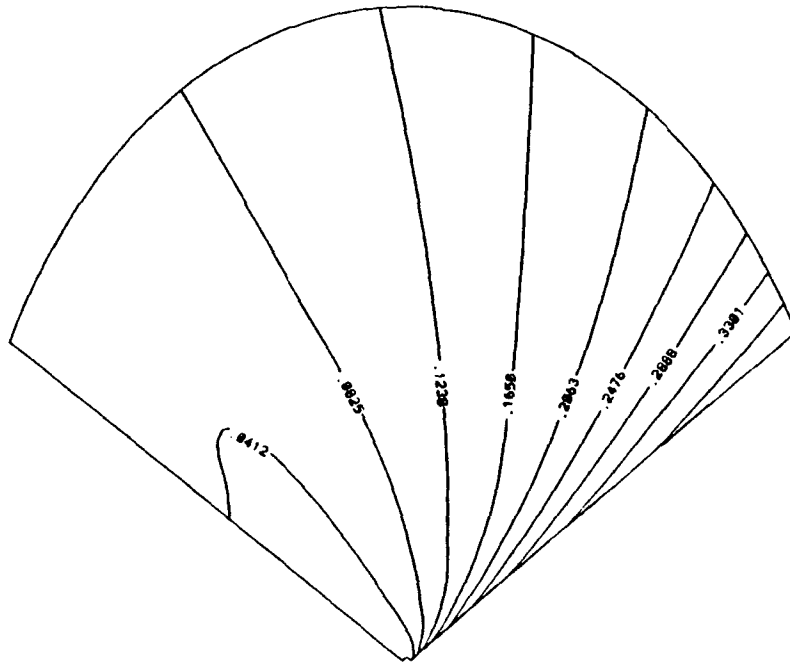
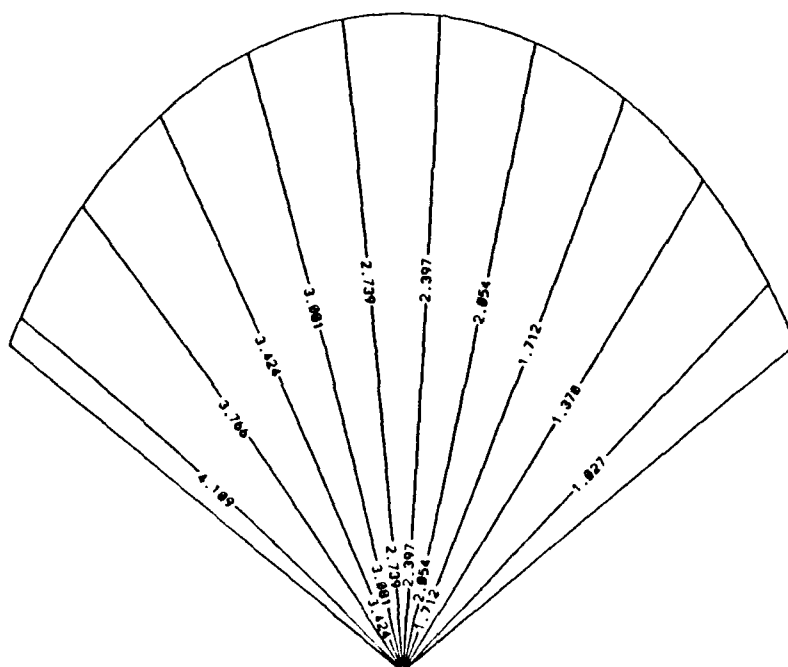


Figure 10.  $TL_{PE}$  at a Depth of 1000 m for Case I with  $\alpha = 0.005$  and  $\beta = 0.1$



**Figure 11. Difference in TL in dB for Case I with  $\alpha = 0.005$  and  $\beta = 0.1$**

The NM and PE transmission losses,  $TL_{NM}$  and  $TL_{PE}$ , at a depth of 1000 m are shown in Figures 18 and 19, respectively. They are nearly the same as can be seen in Figure 20 where the error is everywhere less than 0.2 dB.



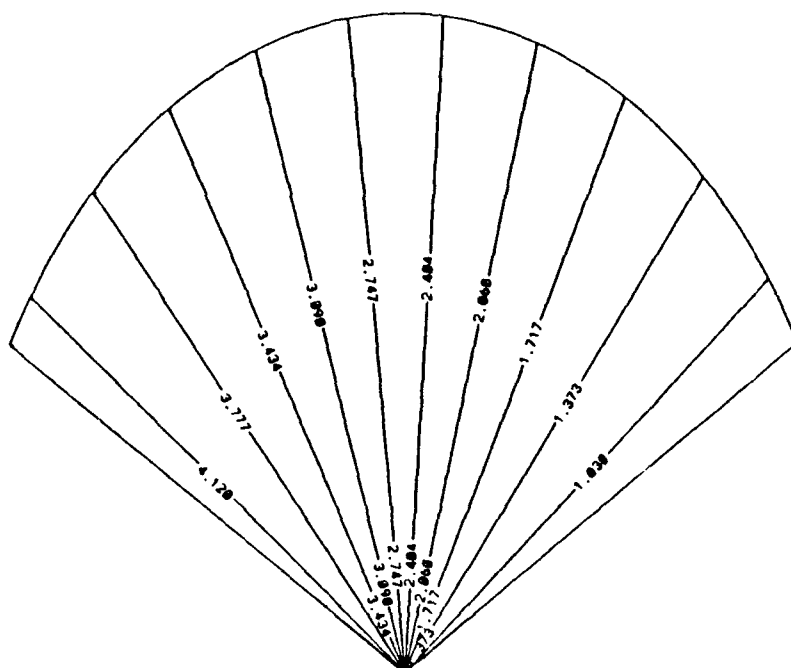


Figure 14. Amplitude of  $U_{PE}$  for Case II with  $\alpha = 4 \times 10^{-7}$  and  $\beta = 0.1$

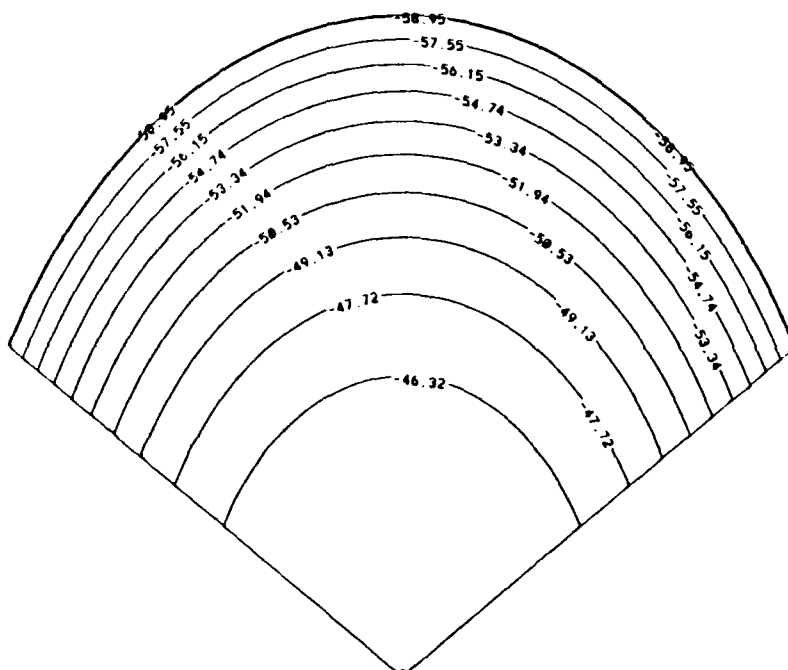


Figure 15. Phase of  $U_{PE}$  for Case II with  $\alpha = 4 \times 10^{-7}$  and  $\beta = 0.1$

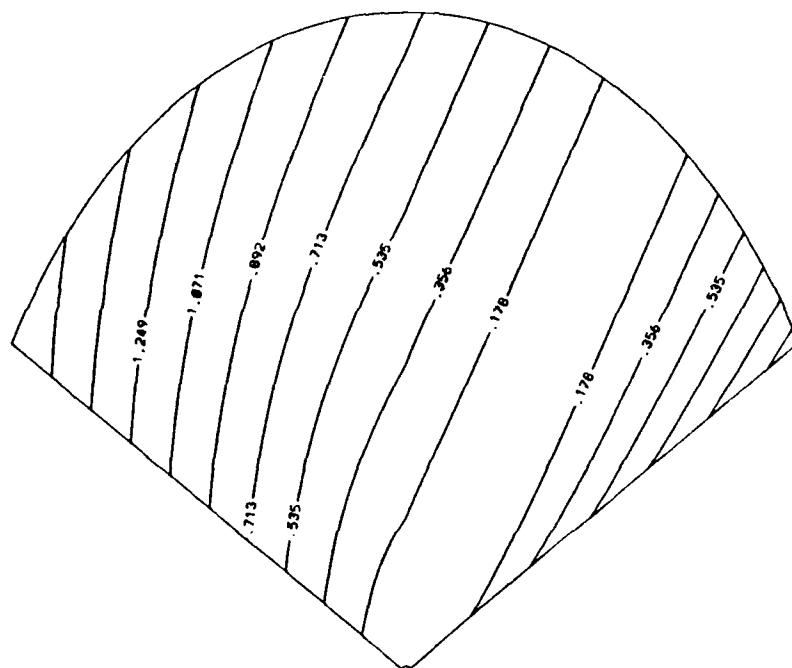


Figure 16. Amplitude Difference in Percent for Case II with  $\alpha = 4 \times 10^{-7}$  and  $\beta = 0.1$

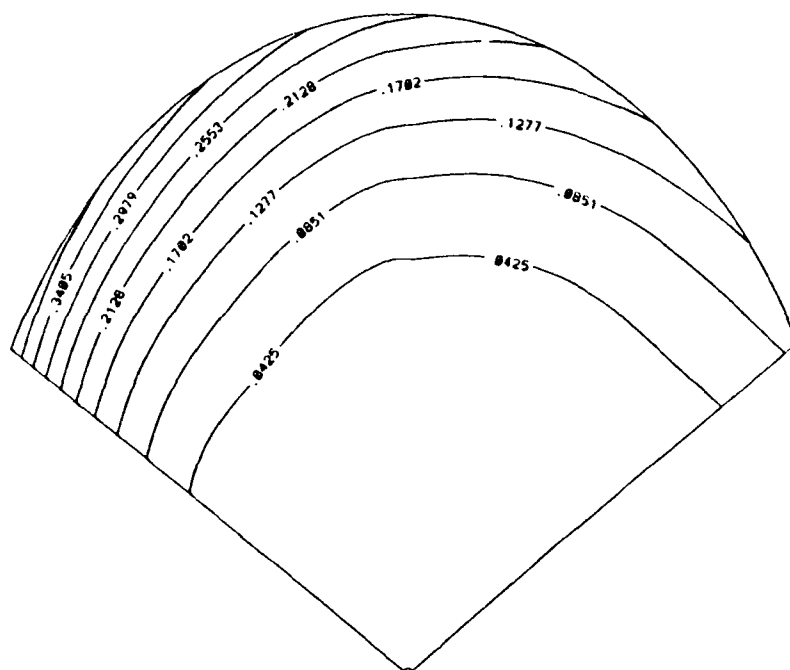


Figure 17. Phase Difference in Degree for Case II with  $\alpha = 4 \times 10^{-7}$  and  $\beta = 0.1$

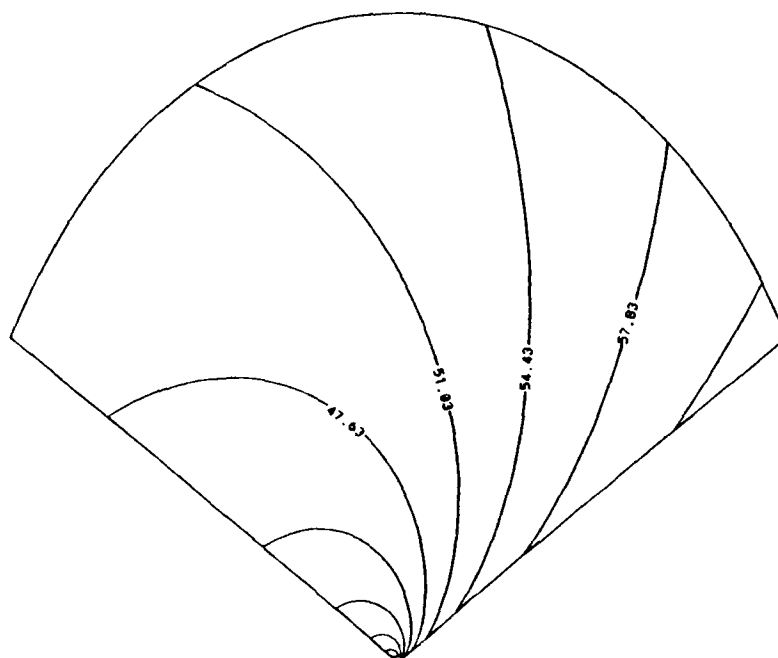


Figure 18.  $TL_{NM}$  at a Depth of 1000 m for Case II with  $\alpha = 4 \times 10^{-7}$  and  $\beta = 0.1$

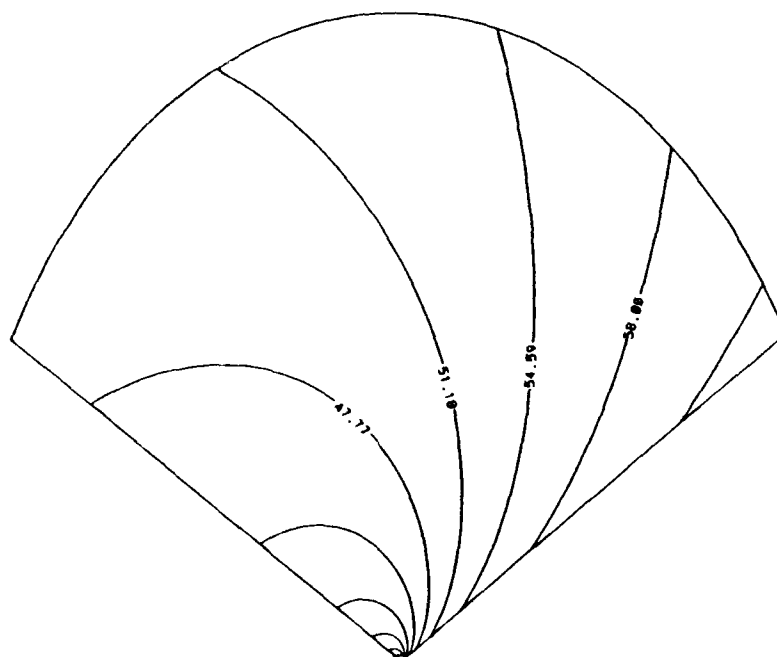


Figure 19.  $TL_{PE}$  at a depth of 1000 m for Case II with  $\alpha = 4 \times 10^{-7}$  and  $\beta = 0.1$

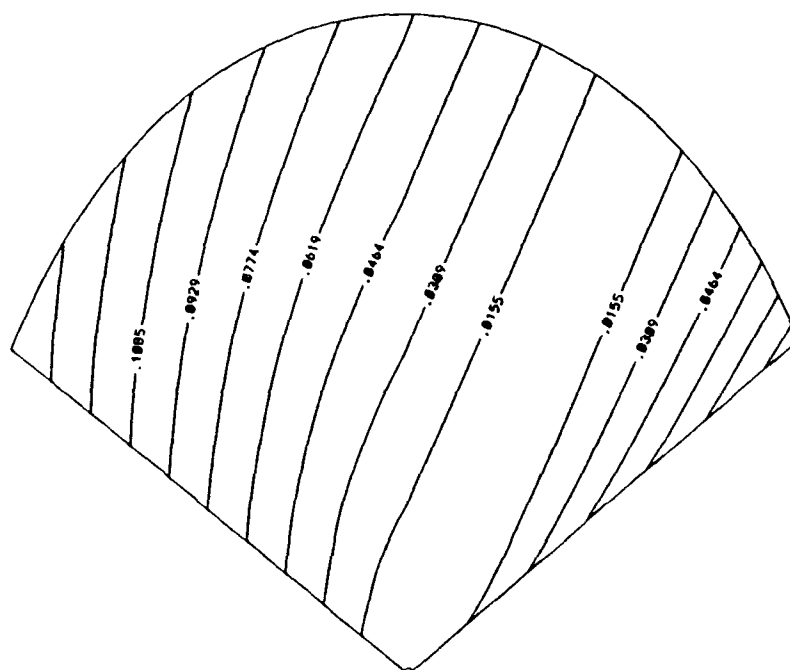


Figure 20. Difference in TL in dB for Case II with  $\alpha = 4 \times 10^{-7}$  and  $\beta = 0.1$

## V. DISCUSSION

In this chapter we address two issues. The first is a quantification of horizontal refraction. The second is an assessment of model accuracy.

### A. HORIZONTAL REFRACTION

Critical to this accuracy test is the use of sound speed fields that cause significant horizontal sound refraction. To confirm that the selected sound speed fields do cause significant horizontal refraction, we compare the corresponding  $N \times 2D$  NM solutions (the calculation is divided into  $N$  vertical slices and a 2D NM model is used in each radial direction) with 3D NM solutions.

The envelope functions of the  $N \times 2D$  solutions have been calculated from (A.25) in the appendix. The amplitude of the NM envelope function is the same for both the  $N \times 2D$  and 3D solutions in Case I and Case II. However, the phases are different. Figure 21 shows the phase of the  $N \times 2D$  envelope function for Case I. The phase is essentially constant at  $-45^\circ$ . In Case II, the phase of the envelope function of the  $N \times 2D$  solution is also constant and its value is also  $-45^\circ$ .

Now let us compare the phases calculated by the  $N \times 2D$  method with the phases of the 3D solutions for Case I and Case II, as displayed in Figures 4 and 13, respectively. Both cases show a  $13^\circ$ - $15^\circ$  phase difference over a range of 100 km, implying that the azimuthal sound speed variation used is large enough to induce sound propagation out of the vertical plane.

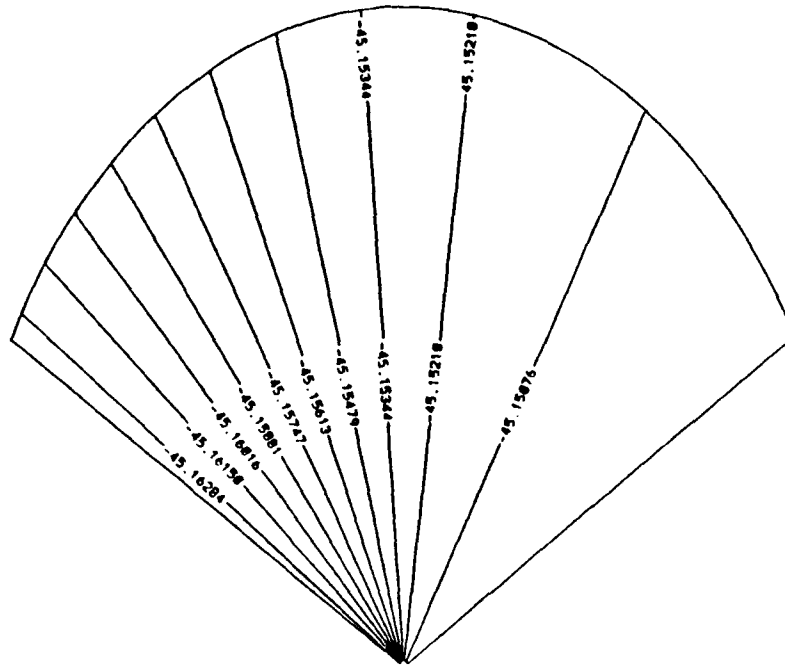


Figure 21. Phase of  $U_{NM}$  for  $N \times 2D$  for Case I with  $\alpha = 0.005$  and  $\beta = 0.1$

## B. ACCURACY

### 1. Phase

In this section we discuss the differences between the analytic PE and numerical NM solutions. In Case I, the phase difference between the analytic PE and numerical NM solutions is less than 2.8 degree everywhere. The Case II phase differences between the numerical NM and analytic PE solutions, as shown in Figure 17, is everywhere less than 1 degree. A quantification of the phase differences for the two runs is given in Table 2.

TABLE 2. PHASE DIFFERENCE

Case	PE		Normal Mode NM		Difference  PE-NM
	min	max	min	max	
Case I	-45.15	-60.35	-45.15	-58.92	2.8
Case II	-45.00	-59.04	-45.00	-58.94	0.43

## 2. Amplitude

Results are shown in Table 3. Envelope amplitude of NM and PE generally agree well with the largest relative difference near 5%. The largest difference occurs in a region that has the largest azimuth gradient in sound speed.

**TABLE 3. AMPLITUDE DIFFERENCE**

Case	Amplitude of NM		Amplitude of PE		Error Value Difference	Relative Difference [%]
	min	max	min	max	max	max
Case I	0.798	4.22	0.817	4.27	0.038	4.8
Case II	0.851	4.28	0.841	4.28	0.076	1.8

## 3. Transmission Loss (TL)

The difference in transmission loss between the analytic PE and numerical NM solutions is shown in Table 4. The difference for both cases is everywhere less than half a decibel.

**TABLE 4. TRANSMISSION LOSS DIFFERENCE**

Case	TL of NM		TL of PE		Difference
	min	max	min	max	max
Case I	28.84	63.31	28.85	63.73	0.43
Case II	28.73	62.75	28.73	62.85	0.15

## VI. CONCLUSIONS

In this work we have tested a three-dimensional numerical acoustic coupled mode model. We have sought to describe its accuracy in regard to the modeling of horizontal refraction. In order to test its accuracy, we compared the coupled mode model results with two different analytic solutions to the parabolic wave equation.

We have concentrated on the accuracy of the envelope function and transmission loss calculation. For the sound speed fields described in Table 1, we found that the phase error of the slowly varying envelope function is lower than 2 degrees and that the envelope amplitude agrees to within 5% or better. Our test cases have indicated that the normal mode model agrees closely with the analytic solutions.

We have only used depth-invariant sound speed fields for this test. As a result, the accuracy in modeling mode-mode interactions is not tested here. Future tests should be directed at examining the accuracy of calculating mode-coupling effects using depth varying sound speed fields.

## APPENDIX

### A. THREE-DIMENSIONAL COUPLED NORMAL MODE METHODS

#### 1. Overview

##### a. *Three-Dimensional (3D) Helmholtz Equation.*

The acoustic pressure  $P(\bar{r}, t)$  in the ocean is governed by the wave equation:

$$\nabla^2 P(\bar{r}, t) - \frac{1}{C^2(\bar{r})} \frac{\partial^2 P(\bar{r}, t)}{\partial t^2} = 0 \quad (\text{A.1})$$

where  $C(\bar{r})$  represents the speed of sound propagation. The cylindrical coordinates are  $\bar{r} = (r, \theta, z)$  where  $r$  is range,  $\theta$  is the azimuthal angle measured positive counterclockwise,  $-z$  is depth, and  $\nabla^2$  is the Laplacian operator in cylindrical coordinates,

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}. \quad (\text{A.2})$$

Inclusion of an acoustic source  $q(\bar{r}, t)$ , modifies Equation (A.1) to

$$\nabla^2 p(\bar{r}, t) - \frac{1}{C^2} \frac{\partial^2 p(\bar{r}, t)}{\partial t^2} = -4\pi q(\bar{r}, t). \quad (\text{A.3})$$

For an acoustic point source located at  $\bar{r}_0$  with time-harmonic circular frequency  $w$ , Equation (A.3) becomes

$$\nabla^2 p(\bar{r}, t) - \frac{1}{C^2} \frac{\partial^2 p(\bar{r}, t)}{\partial t^2} = -4\pi \delta(\bar{r} - \bar{r}_0) e^{-iwt}. \quad (\text{A.4})$$

If we let

$$p(\vec{r}, t) = \Phi(\vec{r})e^{-i\omega t} \quad (\text{A.5})$$

where  $\Phi(\vec{r})$  is time-independent, then a Helmholtz equation results:

$$\nabla^2 \Phi(\vec{r}) + k^2(\vec{r})\Phi(\vec{r}) = -4\pi\delta(\vec{r} - \vec{r}_0) \quad (\text{A.6})$$

where the acoustic wavenumber is  $k(\vec{r}) = \frac{\omega}{c(\vec{r})}$ . For  $\vec{r} \neq \vec{r}_0$ , the Helmholtz Equation is

$$\nabla^2 \Phi(\vec{r}) + k^2(\vec{r})\Phi(\vec{r}) = 0. \quad (\text{A.6a})$$

### b. *Coupled Normal- Mode Solution*

Adiabatic normal mode theory was applied to a range-dependent medium by Pierce [1965] and Milder [1969]. The pressure field was expressed as a linear combination of the local modes whose coefficients were obtained from a system of decoupled ordinary differential equations. The physics of coupled normal mode theory includes non-adiabaticity and results in a *coupled* system of differential equations. The theory used in the development of the Chiu-Ehret 3D coupled normal mode model follows.

The first step in the development is the expansion of the time-independent pressure component in terms of the local normal modes, i.e.,

$$\Phi(\vec{r}) = \frac{1}{\sqrt{r}} \sum_n P_n(r, \theta) Z_n(z; r, \theta) \quad (\text{A.7})$$

where

$Z_n$  is the  $n^{\text{th}}$  local normal modes at point  $(r, \theta)$ .

$P_n$  is the  $n^{\text{th}}$  mode amplitude function at point  $(r, \theta)$ .

### c. *The Local Normal Modes $Z_n$*

The local normal modes obey the following equation:

$$\left[ \frac{\partial^2}{\partial z^2} + (k^2 - k_n^2) \right] Z_n = 0 \quad (\text{A.8})$$

where  $k_n$  is the horizontal wavenumber associated with the  $n^{\text{th}}$  mode at each horizontal location.

The idealized upper and lower boundary conditions are

$$Z_n(z = 0; r, \theta) = 0. \quad (\text{A.9})$$

$$\frac{\partial}{\partial z} Z_n(z = -H; r, \theta) = 0. \quad (\text{A.10})$$

where  $H$  is the ocean depth. The local normal modes can be normalized according to the orthonormal condition

$$\int_0^H Z_n Z_m dz = \delta_{mn}. \quad (\text{A.11})$$

#### d. *The Mode Amplitude Function $P_n$*

To obtain the governing equation for  $P_n$ , substitution of (A.7) in (A.6a) is necessary. The use of (A.8) and the farfield approximation

$$\left[ k_m^2 + \frac{1}{4} \frac{1}{r^2} \right] P_m Z_m \sim k_m^2 P_m Z_m \quad (\text{A.12})$$

following the substitution results in the following equation:

$$\begin{aligned} \sum_m \left[ Z_m \left( \frac{\partial^2 P_m}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 P_m}{\partial \theta^2} \right) + 2 \left( \frac{\partial P_m}{\partial r} \frac{\partial Z_m}{\partial r} + \frac{1}{r^2} \frac{\partial P_m}{\partial \theta} \frac{\partial Z_m}{\partial \theta} \right) \right. \\ \left. + P_m \left( \frac{\partial^2 Z_m}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 Z_m}{\partial \theta^2} \right) + k_m^2 P_m Z_m \right] = 0. \end{aligned} \quad (\text{A.13})$$

The next step is to multiply (A.13) by  $Z_n$  and then integrate over  $z$ , i.e.,

$$\sum_m \left[ \int Z_n Z_m \left( \frac{\partial^2 P_m}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 P_m}{\partial \theta^2} + k_m^2 P_m \right) dz + 2 \int Z_n \left( \frac{\partial P_m}{\partial r} \frac{\partial Z_m}{\partial r} + \frac{1}{r^2} \frac{\partial P_m}{\partial \theta} \frac{\partial Z_m}{\partial \theta} \right) dz + \int P_m Z_n \left( \frac{\partial^2 Z_m}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 Z_m}{\partial \theta^2} \right) dz \right] = 0. \quad (\text{A.14})$$

A subsequent application of the orthonormality condition (A.11) on (A.14) gives

$$\begin{aligned} & \left( \frac{\partial^2}{\partial r^2} + k_n^2 + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) P_n \\ &= - \sum_m \left[ 2 \left( \int Z_n \frac{\partial Z_m}{\partial r} dz \right) \frac{\partial P_m}{\partial r} + \frac{2}{r^2} \left( \int Z_n \frac{\partial Z_m}{\partial \theta} dz \right) \frac{\partial P_m}{\partial \theta} + P_m \int Z_n \left( \frac{\partial^2 Z_m}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 Z_m}{\partial \theta^2} \right) dz \right]. \end{aligned} \quad (\text{A.15})$$

Defining coupling coefficients as

$$\begin{aligned} \gamma_{mn} &= 2 \int Z_n \frac{\partial Z_m}{\partial r} dz \\ \beta_{mn} &= \frac{2}{r} \int Z_n \frac{\partial Z_m}{\partial \theta} dz \\ B_{mn} &= \int Z_n \left( \frac{\partial^2 Z_m}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 Z_m}{\partial \theta^2} \right) dz, \end{aligned}$$

the coupled system (A.15) can be recast as

$$\left( \frac{\partial^2}{\partial r^2} + k_n^2 + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) P_n = - \sum_m \left[ \left( \gamma_{mn} \frac{\partial}{\partial r} + \beta_{mn} \frac{1}{r} \frac{\partial}{\partial \theta} \right) + B_{mn} \right] P_m. \quad (\text{A.16})$$

### *e. The Envelope Functions $U_n$*

Following the work of Chiu and Ehret (1990), the mode amplitude function  $P_n$  can be separated into a slowly varying envelope function  $U_n$  and a rapidly varying component  $e^{i\phi_n(r, \theta)}$ , i.e.,

$$P_n = U_n(r, \theta) e^{i\phi_n(r, \theta)} \quad (\text{A.17})$$

$$\phi_n = \int_0^r k_n(r, \theta) dr. \quad (\text{A.18})$$

A substitution of (A.17) into (A.16) yields

$$\left( \frac{\partial^2}{\partial r^2} + k_n^2 + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) U_n e^{i\phi_n} = - \sum_m \left[ \left( \gamma_{mn} \frac{\partial}{\partial r} + \beta_{mn} \frac{1}{r} \frac{\partial}{\partial \theta} \right) + B_{mn} \right] U_m e^{i\phi_m}$$

which can be expanded as

$$\left[ \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) + i2k_n \frac{\partial}{\partial r} + \left\{ i \frac{\partial k_n}{\partial r} + \frac{i}{r^2} \frac{\partial^2 \phi_n}{\partial \theta^2} - \frac{1}{r^2} \left( \frac{\partial \phi_n}{\partial \theta} \right)^2 \right\} + \frac{2i}{r^2} \frac{\partial \phi_n}{\partial \theta} \frac{\partial}{\partial \theta} \right] U_n e^{i\phi_n} \quad (\text{A.19})$$

$$= - \sum_m \left[ \left( \gamma_{mn} \frac{\partial}{\partial r} + \beta_{mn} \frac{1}{r} \frac{\partial}{\partial \theta} \right) + \left( i\gamma_{mn} k_m + i\beta_{mn} \frac{1}{r} \frac{\partial \phi_m}{\partial \theta} + B_{mn} \right) \right] U_m e^{i\phi_m} \quad (\text{A.20})$$

Dividing (A.20) by  $i2k_n e^{i\phi_n}$  gives the governing equation for the envelope:

$$D_n \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) U_n + \left( \frac{\partial}{\partial r} + E_n \right) U_n + F_n \frac{1}{r} \frac{\partial}{\partial \theta} U_n = \sum_m \bar{G}_{mn} \cdot \nabla U_m + \sum_m H_{mn} U_m \quad (\text{A.21})$$

where

$$D_n = \frac{-i}{2k_n}$$

$$E_n = \frac{1}{2k_n} \left[ \frac{\partial k_n}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi_n}{\partial \theta^2} + \frac{i}{r^2} \left( \frac{\partial \phi_n}{\partial \theta} \right)^2 \right]$$

$$F_n = \frac{1}{k_n r} \left( \frac{\partial \phi_n}{\partial \theta} \right)$$

$$\bar{G}_{mn} = \frac{i}{2k_n} \left[ \gamma_{mn} \hat{r} + \beta_{mn} \hat{\theta} \right] e^{i(\phi_m - \phi_n)}$$

$$\nabla = \frac{\partial}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial}{\partial \theta} \hat{\theta}$$

$$H_{mn} = \begin{cases} \frac{-1}{2k_n} \left[ \gamma_{mn} k_m + \frac{\beta_{mn}}{r} \frac{\partial \phi_m}{\partial \theta} - i B_{mn} \right] e^{i(\phi_m - \phi_n)} & (m \neq n) \\ \frac{i}{2k_n} B_{nn} & (m = n) \end{cases}$$

and  $\hat{r}$  and  $\hat{\theta}$  are the unit directional vectors in  $r$  and  $\theta$ , respectively.

*f. Transmission Loss TL*

The mode amplitude function  $P_n$  is related to the envelope function  $U_n$  and the phase  $\phi_n$  by (A.17). The local normal modes  $Z_n$  are solutions to the eigenvalue-eigenfunction problem (A.8)–(A.11). Once  $P_n$  and  $Z_n$  are computed, pressure can be calculated from the product of  $P_n$  and  $Z_n$ .

At each point  $(r, \theta, z)$ ,  $\Phi^2$  is

$$\Phi^2(r, \theta, z) = [\text{Re}(\Phi)]^2 + [\text{Im}(\Phi)]^2 \quad (\text{A.22})$$

where

$$\text{Re}[\Phi(r, \theta, z)] = \sum_m [\text{Re}(P_m)] \cdot Z_m(z)$$

and

$$\text{Im}(\Phi) = \sum_m [\text{Im}(P_m)] \cdot Z_m(z)$$

In (A.22), Re and Im are used to denote the real and imaginary parts, respectively. Transmission loss can be computed as

$$\text{TL} = -10 \log_{10} \Phi^2(r, \theta, z). \quad (\text{A.23})$$

## 2. Numerical Solution

### a. Numerical Model

To obtain a normal mode solution we must solve the equations for the eigenvalues and eigenmodes (A.8)–(A.11) and the equation for the modulation envelope (A.21). These equations can be solved numerically.

### b. The Eigenvalues $k_n$ and the Eigenmodes $Z_n$

Equation (A.8) is approximated using central finite differencing. This approximation casts (A.8)–(A.10) into a matrix algebra problem. The eigenvalues and the eigenmodes of which can be determined using the iterative QR method (Acton, 1970).

### c. The Iterative Method

To obtain the envelope function  $U_n$ , we need to solve equation (A.21) which can be rearranged to form a set of first order partial differential equations (PDE) with smaller terms put on the right-hand side. We can solve for  $U_n$  in an iterative fashion. The iterative equation is

$$\left[ \frac{\partial}{\partial r} + E_n \right] U_n^i - \sum_m H_{mn} U_m^i = -F_n \frac{1}{r} \frac{\partial}{\partial \theta} U_n^{i-1} - D_n \left[ \frac{\partial^2}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right] U_n^{i-1} + \sum_m \bar{G}_{mn} \cdot \nabla U_n^{i-1} \quad (A.24)$$

where  $U_n^i$  is the solution at the  $i^{th}$  iteration for mode  $n$ . This set of first order differential equations is integrated using a Runge-Kutta method of order 5 and 6 (Acton, 1970).

## 3. The $N \times 2D$ Method

In the  $N \times 2D$  method, the sound channel is divided into  $N$  vertical slices and the two-dimensional normal mode solution is solved for each slice.

It combines the results in each vertical plane to construct a 3D field. This method provides an approximation to a 3D solution for a 3D sound field.

Thus, in each slice the envelope function  $U_n$  is independent of azimuthal variation effects. With the azimuthal terms ignored, the governing equation becomes

$$\left[ \frac{\partial}{\partial r} + E_n \right] U_n^i - \sum_m H_{mn} U_m^i = -D_n \frac{\partial}{\partial r^2} U_n^{i-1} + \sum_m \bar{G}_{mn} \nabla U_m^{i-1} \quad (\text{A.25})$$

where

$$D_n = \frac{-i}{2k_n}$$

$$E_n = \frac{1}{2k_n} \frac{\partial k_n}{\partial r}$$

$$\bar{G}_{mn} = \frac{i}{2k_n} \gamma_{mn} \hat{r} e^{i(\phi_m - \phi_n)}$$

$$\nabla = \frac{\partial}{\partial r} \hat{r}$$

$$H_{mn} = \begin{cases} -\frac{1}{2k_n} (\gamma_{mn} k_m - i B_{mn}) e^{i(\phi_m - \phi_n)} & (m \neq n) \\ \frac{i}{2k_n} B_{nn} & (m = n) \end{cases}$$

## B. THREE-DIMENSIONAL PARABOLIC WAVE EQUATION

### 1. Parabolic Wave Equation (PE)

The following work was developed by Tappert (1977) and Lee (1988).

In cylindrical coordinates, the Helmholtz Equation (A.6a) becomes

$$\frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r} \frac{\partial \Phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \Phi}{\partial \theta^2} + \frac{\partial^2 \Phi}{\partial z^2} + k_0^2 n^2 \Phi(r, \theta, z) = 0 \quad (\text{A.26a})$$

where

$$k_0 = \omega/C_0$$

$C_0$  is a reference sound speed

$\omega$  is the harmonic source frequency

$n$  is the index of refraction

$$n = n(r, \theta, z) = C_0 / C(r, \theta, z).$$

The PE approximation assumes

$$\Phi(r, \theta, z) = u(r, \theta, z)v(r) \quad (A.26b)$$

where  $v(r)$  is rapidly varying and  $u(r, \theta, z)$  is a modulation.

By substituting (A.26b) into (A.26a), one obtains

$$\left[ \frac{\partial^2 u}{\partial r^2} + \left( \frac{1}{r} + \frac{2}{v} \frac{\partial v}{\partial r} \right) \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} + k_0^2 n^2 u \right] v + \left[ \frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} \right] u = 0. \quad (A.26c)$$

If  $v$  is determined by

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} + k_0^2 v = 0 \quad (A.27)$$

then  $u$  must satisfy

$$\frac{\partial^2 u}{\partial r^2} + \left( \frac{1}{r} + \frac{2}{v} \frac{\partial v}{\partial r} \right) \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} + k_0^2 (n^2 - 1) u = 0. \quad (A.28)$$

The solution to (A.27) is

$$v(r) = H_0^1(k_0 r) \sim \left[ \frac{2}{\pi k_0 r} \right]^{\frac{1}{2}} \exp \left[ i \left[ k_0 r - \frac{\pi}{4} \right] \right] \quad (A.29)$$

where  $H_0^1$  is a Hankel function of zeroth order.

By substituting (A.29) into (A.28), one gets

$$\left[ \frac{\partial^2}{\partial r^2} + 2ik_0 \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + k_0^2(n-1) \right] u = 0 \quad (\text{A.30})$$

which can be factored into

$$\left( \frac{\partial}{\partial r} + ik_0 - ik_0 Q \right) \left( \frac{\partial}{\partial r} + ik_0 + ik_0 Q \right) u = 0 \quad (\text{A.31})$$

where

$$Q = \left[ 1 + (n^2 - 1) + \frac{1}{k_0^2} \frac{\partial^2}{\partial z^2} + \frac{1}{k_0^2 r^2} \frac{\partial^2}{\partial \theta^2} \right]^{\frac{1}{2}}.$$

$Q$  is called the square root operator. If one considers only outgoing waves, the equation to be solved is

$$\left( \frac{\partial}{\partial r} + ik_0 - ik_0 Q \right) u = 0. \quad (\text{A.32})$$

Let

$$Q = [1 + X + Y]^{\frac{1}{2}} \quad (\text{A.33})$$

so that

$$X = (n^2 - 1) + \frac{1}{k_0^2} \frac{\partial^2}{\partial z^2}$$

$$Y = \frac{1}{k_0^2 r^2} \frac{\partial^2}{\partial \theta^2}$$

and then expand  $Q$  in a Taylor series:

$$Q = [1 + X + Y]^{\frac{1}{2}} \approx 1 + \frac{1}{2}X - \frac{1}{8}X^2 + \frac{1}{2}Y + \dots \quad (\text{A.34})$$

Inclusion of the first four terms in the expansion leads to a "wide angle" version of the PE approximation. Substituting (A.34) into (A.32) gives

$$\frac{\partial u}{\partial r} = \frac{ik_0}{2} \left[ \left( n^2 - 1 \right) + \frac{1}{k_0^2} \frac{\partial^2}{\partial z^2} - \frac{1}{4} \left\{ \left( n^2 - 1 \right) + \frac{1}{k_0^2} \frac{\partial^2}{\partial z^2} \right\}^2 + \frac{1}{k_0^2 r^2} \frac{\partial^2}{\partial \theta^2} \right] u. \quad (\text{A.35})$$

## 2. The Envelope Function $U_p$ of the Parabolic Equation

Here we present the analytical results of Seigmann et al., (1990). Their development uses the pressure release surface and rigid bottom boundary conditions, i.e.,

$$\begin{aligned} u(r, \theta, z = 0) &= 0 \\ \frac{\partial u}{\partial z}(r, \theta, z = H) &= 0. \end{aligned} \quad (\text{A.36})$$

A solution to equation (A.35) for a depth-independent sound speed field and subject to the boundary conditions stated above can take the following form:

$$u(r, \theta, z) = U_p(r, \theta) \sin(\gamma_j z) \exp \left\{ - \frac{ik_0 r}{2} \left( \frac{\gamma_j}{k_0} \right)^2 \left[ 1 - \frac{1}{4} \left( \frac{\gamma_j}{k_0} \right)^2 \right] \right\} \quad (\text{A.37})$$

where

$$\gamma_j = \left( j + \frac{1}{2} \right) \frac{\pi}{H},$$

$H$  is the ocean depth, and  $U_p$  is the envelope function of the parabolic solution.

By substituting (A.37) into (A.35), one obtains the equation governing  $U_p$ :

$$\frac{\partial U_p(r, \theta)}{\partial r} = \frac{ik_0}{2} \left\{ (n^2 - 1) \left[ 1 + \frac{1}{2} \left( \frac{\gamma_i}{k_0} \right)^2 \right] - \frac{1}{4} (n^2 - 1)^2 - \frac{1}{2} \left( \frac{\gamma_j}{k_0} \right)^4 + \frac{1}{(k_0 r)^2} \frac{\partial^2}{\partial \theta^2} \right\} U_p(r, \theta). \quad (\text{A.38})$$

Letting

$$U_p(r, \theta) = A_p(r, \theta) e^{i\Theta(r, \theta)} \quad (\text{A.39})$$

where both  $A_p(r, \theta)$  and  $\Theta(r, \theta)$  are real quantities, (A.38) becomes

$$\begin{aligned} \frac{\partial A_p}{\partial r} + i A_p \frac{\partial \Theta}{\partial r} = \frac{ik_0 A_p}{2} \left\{ (n^2 - 1) \left[ 1 + \frac{1}{2} \left( \frac{\gamma_j}{K_0} \right)^2 \right] - \frac{1}{4} (n^2 - 1)^2 - \frac{1}{2} \left( \frac{\gamma_j}{k_0} \right)^4 \right\} \\ + \frac{i}{2k_0 r^2} \left[ \frac{\partial^2 A_p}{\partial \theta^2} + 2i \frac{\partial A_p}{\partial \theta} \frac{\partial \Theta}{\partial \theta} + i A_p \frac{\partial^2 \Theta}{\partial \theta^2} - \left( \frac{\partial \Theta}{\partial \theta} \right)^2 A_p \right] \end{aligned} \quad (\text{A.40})$$

A separation of the real and imaginary parts of (A.40) yields

$$\frac{\partial A_p}{\partial r} + \frac{1}{k_0 r^2} \frac{\partial A_p}{\partial \theta} \frac{\partial \Theta}{\partial \theta} + \frac{A_p}{2k_0 r^2} \frac{\partial^2 \Theta}{\partial \theta^2} = 0 \quad (\text{A.41})$$

$$(n^2 - 1) \left[ 1 + \frac{1}{2} \left( \frac{\gamma_j}{k_0} \right)^2 \right] - \frac{1}{4} (n^2 - 1)^2 - \frac{1}{2} \left( \frac{\gamma_j}{k_0} \right)^4 = \frac{2}{k_0} \frac{\partial \Theta}{\partial r} + \frac{1}{(k_0 r)^2} \left( \frac{\partial \Theta}{\partial \theta} \right)^2 - \frac{1}{(k_0 r)^2} \frac{1}{A_p} \frac{\partial^2 A_p}{\partial \theta^2} \quad (\text{A.42})$$

Given  $A_p$  and  $\Theta$ , a pressure field is determined, and the index of refraction can be calculated from (A.42). The resultant sound speed field can then be used in the normal mode model.

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